

BROWN'S LEMMA IN SECOND-ORDER ARITHMETIC

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ABSTRACT. Brown's lemma states that in every finite coloring of the natural numbers there is a homogeneous piecewise syndetic set. We show that Brown's lemma is equivalent to IS_2^0 over RCA_0^* . We show in contrast that (the infinite) van der Waerden's theorem is equivalent to BS_2^0 over RCA_0^* . We finally consider the finite version of Brown's lemma and show that it is provable in RCA_0 but not in RCA_0^* .

1. INTRODUCTION

In the present paper we study Brown's lemma in the context of second-order arithmetic. Brown's lemma asserts that piecewise syndetic sets are large, in the sense that whenever we partition the natural numbers into finitely many sets, at least one set must be piecewise syndetic.

Definition 1.1 (Piecewise syndetic). A set $X \subseteq \mathbb{N}$ is *piecewise syndetic* if there exists $d \in \mathbb{N}$ such that X contains arbitrarily large finite sets with gaps bounded by d , where a set has gaps bounded by d if the difference between any two consecutive elements is $\leq d$. We also say that X is piecewise d -syndetic if d witnesses that X is piecewise syndetic.

Theorem 1.2 (Brown's Lemma). *Every finite coloring $C: \mathbb{N} \rightarrow r$ has a C -homogeneous piecewise syndetic set.*

Note that piecewise syndetic sets are closed under supersets and so we can reformulate Brown's lemma as follows.

Theorem 1.3 (Brown's lemma, restated). *Let $\mathbb{N} = X_0 \cup \dots \cup X_{r-1}$ be a finite partition of the natural numbers. Then X_i is piecewise syndetic for some $i < r$.*

In fact, Brown's lemma generalizes to finite partitions of piecewise syndetic sets. Recall that a class of sets $\mathbb{S} \subseteq \mathcal{P}(\mathbb{N})$ is called *partition regular* if for every $X \in \mathbb{S}$ and for every finite partition $X = X_0 \cup \dots \cup X_{r-1}$ there exists $i < r$ such that $X_i \in \mathbb{S}$.

Theorem 1.4 (Partition regularity of piecewise syndetic sets). *Let $X \subseteq \mathbb{N}$ be piecewise syndetic and $X = X_0 \cup \dots \cup X_{r-1}$ be a finite partition. Then X_i is piecewise syndetic for some $i < r$.*

Theorem 1.2 was originally proved by Brown [Bro68] (see also [Bro69, Bro71]) in the context of locally finite semigroups. An ergodic-theoretic proof of Theorem 1.2 as well as Theorem 1.4 can be found in Furstenberg [Fur81, Theorem 1.23, Theorem 1.24]. For an algebraic-theoretic proof of Theorem 1.4 based on the characterization of piecewise syndetic sets in terms of $\beta\mathbb{N}$, the Stone-Ćech compactification of \mathbb{N} , see Hindman [HS12].

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As piecewise syndetic sets are arithmetically (Σ_3^0)-definable and closed under supersets, Brown's lemma does not actually assert the existence of a set. In other words, this is a Π_1^1 -statement. Notice that a partition lemma of the form

† Every finite coloring $C: \mathbb{N} \rightarrow r$ has a C -homogeneous *large* set,

where *large* is a property about sets closed under supersets, is computably true. If *large* is also arithmetical, then † is likely to be provable in $\text{RCA}_0 + \text{IS}_n$ if *large* is Σ_{n+1}^0 -definable and in $\text{RCA}_0 + \text{BS}_n^0$ if *large* is Π_n^0 -definable. For instance the infinite pigeonhole principle

$\text{RT}_{<\infty}^1$ Every finite coloring $C: \mathbb{N} \rightarrow r$ has an infinite C -homogeneous set, is provable and actually equivalent to BS_2^0 over RCA_0 . Therefore $\text{RT}_{<\infty}^1$ and Brown's lemma serve as an example for $n = 2$.

Brown's lemma is related to the well-known van der Waerden's theorem.

Theorem 1.5 (Van Der Waerden's Theorem). *Every finite coloring $C: \mathbb{N} \rightarrow r$ has a C -homogenous set with arbitrarily long arithmetic progressions.*

Say that $X \subseteq \mathbb{N}$ is AP if it contains arbitrarily long arithmetic progressions. Note that AP sets are also closed under supersets and so we can restate van der Waerden's theorem as follows.

Theorem 1.6 (Van Der Waerden's Theorem, restated). *Let $\mathbb{N} = X_0 \cup \dots \cup X_{r-1}$ be a finite partition of the natural numbers. Then X_i is AP for some $i < r$.*

Also van der Waerden's theorem generalizes to finite partitions of AP sets.

Theorem 1.7 (Partition regularity of AP sets). *Let $X \subseteq \mathbb{N}$ be AP and $X = X_0 \cup \dots \cup X_{r-1}$ be a finite partition. Then X_i is AP for some $i < r$.*

It is known that every piecewise syndetic set contains arbitrarily long arithmetic progressions and so Brown's lemma is a generalization of van der Waerden's theorem. On the other hand, AP sets need not be piecewise syndetic and so partition regularity for piecewise syndetic sets does not imply partition regularity for AP sets. The proof that piecewise syndetic sets are AP uses the finite version of van der Waerden's theorem (see Rabung [Rab75]).

Theorem 1.8 (Van Der Waerden's Theorem, finite). *For all r, l there exists n such that every coloring $C: n \rightarrow r$ has a C -homogeneous arithmetic progression of length l .*

Let $W(r, l)$ be the least number n such that every r -coloring of n has a homogeneous arithmetic progression of length l . By Shelah [She88] the function $W(l, r)$ has primitive recursive upper bounds. These bounds were obtained as a byproduct of a new proof of Hales-Jewett theorem which avoids the use of double induction and can be formalized in RCA_0 .¹

Theorem 1.9 (Folklore). *The finite version of van der Waerden's theorem is provable in RCA_0 .*

¹All the previous proofs of Hales-Jewett theorem used double induction giving Ackermannian upper bounds. Gowers [Gow98, Gow01] in his celebrated work on Szemerédi's theorem obtained elementary recursive upper bounds for the van der Waerden's numbers. Gowers' bound is the following:

$$W(r, l) \leq 2^{2^{f(r, l)}}, \text{ where } f(r, l) = r^{2^{l+9}}.$$

However, the proof is far from elementary and so we can only conjecture that van der Waerden's theorem is provable in EFA. On the other hand, the lower bounds for the van der Waerden's numbers are exponential (see [GRS90]), and hence van der Waerden's theorem is not provable in bounded arithmetic.

Working in the framework of reverse mathematics we prove that Brown's lemma is equivalent to $\text{I}\Sigma_2^0$ over RCA_0^* and so is the statement that piecewise syndetic sets are partition regular. We also show that van der Waerden's theorem and partition regularity for AP sets are equivalent to $\text{B}\Sigma_2^0$ over RCA_0^* .

We finally consider the finite version of Brown's lemma.

Definition 1.10. Let $H \subseteq \mathbb{N}$ be finite. Define the *gap size* of H , denoted $gs(H)$, as the largest difference between two consecutive elements of H . In other words, the gap size of H is the least d such that H has gaps bounded by d . If $|H| \leq 1$ let $gs(H) = 1$.

Theorem 1.11 (Brown's Lemma, finite). *Let $f: \mathbb{N} \rightarrow \mathbb{N}$. Then for all $r > 0$ there exists n such that every coloring $C: n \rightarrow r$ has a C -homogeneous set H with $|H| > f(gs(H))$.*

The finite version of Brown's lemma is reminiscent of the Paris-Harrington principle. We can think of $|H| > f(gs(H))$ as a largeness condition on H .

Definition 1.12. For $f: \mathbb{N} \rightarrow \mathbb{N}$, let BL_f be the statement "For all $r > 0$ there exists n such that every $C: n \rightarrow r$ has a C -homogeneous set H such that $|H| > f(gs(H))$."

We show that the finite version of Brown's lemma is provable in RCA_0 but not in RCA_0^* . We obtain the latter result by showing that BL_f is not provable in EFA for $f(d) = 2^d$. On the other hand, BL_f is provable in EFA for $f(d) = d$. Therefore the finite version of Brown's lemma provides a natural example of phase transition with respect to EFA .

In the appendix we discuss the original proof of Brown's lemma and point out a disguised use of ACA_0 .

2. PRELIMINARIES

We assume familiarity with the basic systems of reverse mathematics. The standard reference is [Sim09]. In this paper we are mainly concerned with the base systems RCA_0^* , RCA_0 and the induction schemes $\text{I}\Sigma_2^0$ and $\text{B}\Sigma_2^0$. Recall that RCA_0^* consists of the usual axioms for addition, multiplication, exponentiation plus Σ_0^0 -induction and Δ_1^0 -comprehension. The system RCA_0 consists of the usual axioms for addition and multiplication plus Σ_1^0 -induction and Δ_1^0 -comprehension. We have $\text{RCA}_0 = \text{RCA}_0^* + \text{I}\Sigma_1^0$. We use the following fact repeatedly.

Theorem 2.1 (see Hirst [Hir87] and Yokoyama [Yok13]). *Over RCA_0^* , $\text{RT}_{<\infty}^1$ is equivalent to $\text{B}\Sigma_2^0$.*

Simpson and Smith [SS86] proved that the first-order part of WKL_0^* , that is RCA_0^* plus Weak König's Lemma, is the same as that of $\text{B}\Sigma_1 + \text{exp}$ and that WKL_0^* is Π_2 -conservative over $\text{I}\Sigma_0 + \text{exp}$, also known as EFA (Elementary Function Arithmetic). Recall that the provably recursive functions of EFA are exactly the elementary recursive functions and that every elementary recursive function $f(n)$ is dominated by $2_k(n)$ for some number k , where $2_k(n)$ is the superexponential function defined by $2_0(n) = n$ and $2_{k+1}(n) = 2^{2_k(n)}$ (see for instance [SW12]).

3. PIECEWISE SYNDETIC SETS

In the present paper we do not consider the characterization of piecewise syndetic sets in terms of $\beta\mathbb{N}$ (cf. Hindman [HS12, Theorem 4.40]). The purpose of this section is to show that, with few exceptions, most of the elementary characterizations for piecewise syndetic sets can be proved within RCA_0^* .

Definition 3.1. An infinite set $X \subseteq \mathbb{N}$ is *syndetic* if there exists $d \in \mathbb{N}$ such that X has gaps bounded by d . We say that X is d -syndetic if d is such a witness. A set $X \subseteq \mathbb{N}$ is *thick* if it contains arbitrarily long intervals of natural numbers.

Proposition 3.2 (RCA_0^*). *Let $X \subseteq \mathbb{N}$. The following are equivalent:*

- (1) X is piecewise syndetic;
- (2) X is the intersection of a syndetic set and a thick set.

Proof. We argue in RCA_0^* . Implication (2) \rightarrow (1) is straightforward. For (1) \rightarrow (2) suppose X is piecewise d -syndetic. Let $Z = \bigcup_{s < d} X + s = \bigcup_{s < d} \{x + s : x \in X\}$. We claim that Z is thick. Fix n and let H be an n -element subset of X such that $gs(H) \leq d$. Then $I = \bigcup_{s < d} H + s \subseteq Z$ is an interval of size at least n . Define $Y = X \cup (\mathbb{N} \setminus Z)$. Clearly $X = Y \cap Z$. It suffices to show that Y is syndetic. Suppose not. Then $A = \mathbb{N} \setminus Y$ is thick and $A \subseteq Z$. In particular $X \cap A = \emptyset$. Let $I \subseteq A$ be an interval of size at least d . Every element of I is of the form $x + s$ with $x \in X$ and $s < d$. It follows that $X \cap I \neq \emptyset$ since otherwise $|I| < d$, and so $X \cap A \neq \emptyset$, a contradiction. \square

Definition 3.3. We say that $I \subseteq X$ is an *interval* of X if for all $x < y$ in I we have $(x, y) \cap X = I$.

Proposition 3.4. *Let $X \subseteq \mathbb{N}$. The following are equivalent:*

- (1) There exists d such that $gs(H) \leq d$ for arbitrarily large finite sets $H \subseteq X$, i.e. X is piecewise syndetic;
- (2) There exists d such that $gs(H) = d$ for arbitrarily large finite sets $H \subseteq X$;
- (3) There exists d such that $gs(I) \leq d$ for arbitrarily long intervals I of X ;
- (4) There exists d such that $gs(I) = d$ for arbitrarily long intervals I of X .

Most implications are trivial and provable in RCA_0^* . The only nontrivial implications are (1) \rightarrow (2) and (1) \rightarrow (4).

Proposition 3.5. *Over RCA_0^* , $\text{B}\Sigma_2^0$ is equivalent to (1) \rightarrow (4).*

Proof. We work in $\text{RCA}_0 + \text{B}\Sigma_2^0$ and assume (1). Suppose X is piecewise d -syndetic. For all n , search for an n -element interval I_n of X such that $gs(I_n) \leq d$. Define a coloring $C: \mathbb{N} \rightarrow d+1$ by letting $C(n) = gs(I_n)$. By $\text{RT}_{<\infty}^1$ there exists $e \leq d$ such that $\{n: C(n) = e\}$ is infinite. Then e is as desired.

We work in RCA_0^* and assume (1) \rightarrow (4). We aim to show $\text{RT}_{<\infty}^1$. Let $C: \mathbb{N} \rightarrow r$. For convenience we assume $C(n) > 0$ for all n . We define a piecewise r -syndetic set $X = \{x_0 < x_1 < \dots\} \subseteq \mathbb{N}$ such that for every $n > 0$ there is an interval I_n of X of size n with $gs(I_n) = C(n) < r$ and $\min(I_{n+1}) - \max(I_n) = n$.

$$1 \underbrace{10 \dots 01}_{|I_2|=2} 0 \underbrace{10 \dots 01}_{|I_3|=3} 0 \underbrace{10 \dots 01}_{|I_3|=3} 001 \dots$$

$\begin{array}{ccc} C(2)-1 \text{ times} & C(3)-1 \text{ times} & C(3)-1 \text{ times} \\ \hline \end{array}$

Let $x_0 = 0$, $x_1 = x_0 + 1$ and $x_2 = x_1 + C(2)$, $x_3 = x_2 + 2$, $x_4 = x_3 + C(3)$ and $x_5 = x_4 + C(3)$. In general, suppose we have defined x_k and $k+1 = n(n+1)/2 = 1 + 2 + 3 + \dots + n$. Then

$$\begin{aligned} x_{k+1} &= x_k + n; \\ x_{k+i+1} &= x_{k+i} + C(n+1) \text{ for } 0 < i < n+1. \end{aligned}$$

Since $x_k \leq rk(k+1)/2$, we can define x_k by bounded primitive recursion and X by Δ_1^0 -comprehension. By construction, $I_n = \{x_k, x_{k+1}, \dots, x_{k+n-1}\}$, where $k = (n-1)n/2$, is an interval of X of size n and gaps bounded by r , and hence X is piecewise r -syndetic. By (1) \rightarrow (4), there exists d such that $gs(I) = d$ for arbitrarily long intervals I of X . We claim that $d < r$ and $\{n \in \mathbb{N} : C(n) = d\}$ is infinite. Suppose for a contradiction that $d \geq r$ or there exists n such that $C(m) \neq d$ for all $m > n$. In both cases there exists $n > d$ such that $C(m) \neq d$ for all $m > n$. Let $k+1 = 2+3+\dots+n$. It follows that if I is a 2-element interval of X such that $gs(I) = d$ then $I \subseteq x_k + 1$ and hence $|I| \leq x_k + 2$, against our assumption on d . \square

The reversal in the proof of Theorem 3.5 does not work for (1) \rightarrow (2). In fact, if the set $\{x \in \mathbb{N} : C(x) = d\}$ is infinite, then for every multiple e of d there are arbitrarily large subsets H of X with $gs(H) = e$.

Question. What is the strength of (1) \rightarrow (2)? By Theorem 3.5, (1) \rightarrow (2) is provable in $\text{RCA}_0 + \text{B}\Sigma_2^0$.

4. BROWN'S LEMMA VS VAN DER WAERDEN'S THEOREM

The proof that every piecewise syndetic set contains arbitrarily long arithmetic progressions is based on the finite version of van der Waerden's theorem and can be formalized in RCA_0^* . Therefore it is not surprising that within RCA_0 Brown's lemma implies van der Waerden's theorem. Here we establish this fact by showing that van der Waerden's theorem is equivalent to $\text{B}\Sigma_2^0$ over RCA_0^* .

Theorem 4.1. *Over RCA_0^* , the following are equivalent:*

- (1) $\text{B}\Sigma_2^0$;
- (2) *Van der Waerden's theorem*;
- (3) *Partition regularity of AP sets*.

Proof. We argue in RCA_0^* . Clearly (3) \rightarrow (2). As van der Waerden's theorem implies $\text{RT}_{<\infty}^1$, which is equivalent to $\text{B}\Sigma_2^0$, we have (2) \rightarrow (1). It remains to show (1) \rightarrow (3). Let $X = X_0 \cup \dots \cup X_{r-1}$ be a finite partition of an AP set. Suppose for a contradiction that X_i is not AP for all $i < r$. Then for all $i < r$ there exists l such that no arithmetic progression of length l lies within X_i . By $\text{B}\Sigma_2^0$ there exists l large enough such that for all $i < r$ no arithmetic progression of length l lies within X_i . By the finite van der Waerden's theorem, let n be such that every coloring $C: n \rightarrow r$ has a C -homogeneous arithmetic progression of length l . Let $\{x_0 < x_1 < \dots < x_{n-1}\} \subseteq X$ be an arithmetic progression of length n . Define a coloring $C: n \rightarrow r$ by letting $C(m) = i$ iff $x_m \in X_i$. Then there exists a C -homogeneous arithmetic progression $m_0 < m_1 < \dots < m_{l-1}$ of length l . It follows that $x_{m_0} < x_{m_1} < \dots < x_{m_{l-1}}$ is an arithmetic progression of length l which lies entirely within X_i for some $i < r$, for the desired contradiction. \square

Notice that we use the finite van der Waerden's theorem in the course of the proof.

Corollary 4.2. *Over RCA_0^* , Brown's lemma implies van der Waerden's theorem.*

Proof. Over RCA_0^* Brown's lemma implies $\text{RT}_{<\infty}^1$, which is equivalent to $\text{B}\Sigma_2^0$. \square

5. BROWN'S LEMMA

We first show that Brown's lemma is provable in IS_2^0 . The argument is due to Kreuzer [Kre12].

Lemma 5.1 (RCA_0^*). *If $X \subseteq \mathbb{N}$ is d -syndetic and $X = X_0 \cup X_1$ is a 2-partition, then either each X_i is piecewise d -syndetic or one of them is syndetic.*

Proof. Suppose X_0 is not syndetic. This means that for arbitrarily long intervals I of \mathbb{N} we have $X_0 \cap I = \emptyset$ and so $X \cap I = X_1 \cap I$. Therefore X_1 is piecewise d -syndetic. \square

Theorem 5.2. *Over RCA_0 , IS_2^0 implies Brown's Lemma.*

Proof. Let $C: \mathbb{N} \rightarrow r$ be a finite coloring. By bounded Σ_2^0 -comprehension let

$$I = \{A \subseteq r: \bigcup_{i \in A} \{x \in \mathbb{N}: C(x) = i\} \text{ is syndetic}\}.$$

I is nonempty as $r \in I$. Let $A \in I$ be minimal (w.r.t. \subseteq). Notice that $A \neq \emptyset$. Suppose that the union is d -syndetic. By Lemma 5.1 and by the minimality of I , for every $i \in A$ the set $\{x \in \mathbb{N}: C(x) = i\}$ must be piecewise d -syndetic. \square

Note that the proof of Theorem 5.2 does not generalize to partitions of piecewise syndetic sets. However we have the following:

Theorem 5.3. *Over RCA_0^* , the following are equivalent:*

- (1) *Brown's lemma;*
- (2) *Partition regularity of piecewise syndetic sets.*

Proof. We argue in RCA_0^* and assume Brown's lemma. In particular we have Σ_1^0 -induction. Let $X = X_0 \cup X_1 \dots \cup X_{r-1}$ be a finite partition of X and suppose that X is piecewise d -syndetic. By primitive recursion we can define a piecewise d -syndetic subset $\{x_0 < x_1 < x_2 < \dots\}$ of X such that for every $n > 0$ the n -element subset $I_n = \{x_k, x_{k+1}, x_{k+2}, \dots, x_{k+n-1}\} \subseteq X$ has gaps bounded by d , where $k = 1 + 2 + \dots + (n - 1)$. Define $C: \mathbb{N} \rightarrow r$ by letting $C(n) = i$ iff $x_n \in X_i$. By Brown's Lemma, there exists a C -homogeneous piecewise syndetic set Y . Say Y is piecewise e -syndetic and homogeneous for color i .

We claim that $Z = \{x_n: n \in Y\} \subseteq X_i$ is piecewise $e \cdot d$ -syndetic. To this end it is enough to show that for every n there exists an n -element subset A of Y such that $gs(A) \leq e$ and $\{x_n: n \in A\} \subseteq I_m$ for some m . Let $n > 0$ be given. Set $k = 1 + 2 + \dots + (2ne - 1)$ so that $x_k \in B_{2ne}$. Now consider $p = k + 2n$. As Y is piecewise e -syndetic, there exists a p -element subset $\{i_0 < \dots < i_{p-1}\}$ of Y with gaps bounded by e . Let $A = \{i_k < i_{k+1} < \dots < i_{p-1}\}$. Notice that $|A| = 2n$ and $gs(A) \leq e$. Since $i_k \geq k$ we have that $x_{i_k} \in I_m$ with $|I_m| \geq 2ne$. As $gs(A) \leq e$ we have that $i_{p-1} - i_k \leq 2ne$ and so $|\{x_i: i_k \leq i \leq i_p\}| \leq 2ne + 1$. It follows that $\{x_i: i_k \leq i \leq i_p\} \subseteq I_m \cup I_{m+1}$. This ensures that $\{x_i: i \in A\} \subseteq I_m \cup I_{m+1}$. Then either the first n elements are in I_m or the last n elements are in I_{m+1} . In both cases we are done. \square

We now turn to the reversal. Actually we show that the following weak version of Brown's lemma already implies IS_2^0 over RCA_0^* .

Theorem 5.4 (Weak Brown's Lemma). *For every coloring $C: \mathbb{N} \rightarrow r$ there exists $d \in \mathbb{N}$ such that $gs(H) \leq d$ for arbitrarily large C -homogeneous sets H .*

Theorem 5.5. *Over RCA_0^* , the following are equivalent:*

- (1) IS_2^0 ;
- (2) *Brown's lemma*;
- (3) *Partition regularity of piecewise syndetic sets*;
- (4) *Weak Brown's lemma*.

We need the following diagonalization lemma.

Lemma 5.6 (RCA_0^*). *There exists a function $D: \mathbb{N} \times \mathbb{N} \rightarrow 2$ such that for all d the 2-coloring $D(d, \cdot)$ has no arbitrarily large C -homogeneous sets H with $gs(H) \leq d$.*

Proof. For all $d > 0$ let

$$D(d, \cdot) = \overbrace{0 \dots 0}^{d \text{ times}} \overbrace{1 \dots 1}^{d \text{ times}} \overbrace{0 \dots 0}^{d \text{ times}} \overbrace{1 \dots 1}^{d \text{ times}} \dots$$

We define D by $D(d, x) = \lfloor \frac{x}{d} \rfloor \pmod{2}$. Fix $d > 0$ and let $C(x) = D(d, x)$. Suppose H is C -homogeneous with gaps bounded by d . We claim that $|H| \leq d$. Let $H = \{x_0 < x_1 < \dots < x_l\}$ and $m = \lfloor \frac{x_0}{d} \rfloor$ so that $md \leq x_0 < (m+1)d$. We show by induction that $x_l < (m+1)d$ for all l . The case $l = 0$ is true. Suppose $x_l < (m+1)d$. Since $x_{l+1} - x_l \leq d$ we have that $x_{l+1} < (m+2)d$. If $(m+1)d \leq x_{l+1}$ then $x_{l+1}/d = m+1$ and $C(x_{l+1}) = m+1 \pmod{2} \neq m \pmod{2} = C(x_l)$, against H being homogeneous. Therefore $H \subseteq [md, (m+1)d)$ and hence $|H| \leq d$. \square

Proof of Theorem 5.5. Implication (1) \rightarrow (2) is Theorem 5.2 and (2) \leftrightarrow (3) is Theorem 5.3. Clearly (2) \rightarrow (3). It remains to show (3) \rightarrow (1).

Within IS_0^0 , one can show that IS_{n+1}^0 is equivalent to SII_n^0 , strong collection for Π_n^0 -formulas (see [HP93, Ch. I Sect. 2(b)]). We first assume IS_1^0 and prove SII_1^0 , that is:

$$(\forall a)(\exists d)(\forall x < a)(\exists y \forall z \theta(x, y, z) \rightarrow (\exists y < d)(\forall z) \theta(x, y, z)),$$

where θ is Σ_0^0 . Let $f(x, s)$ be the least $y \leq s$ such that $(\forall z < s) \theta(x, y, z)$, and s if there is no such a y . By Σ_1^0 -induction one can show that if $(\exists y)(\forall z) \theta(x, y, z)$ then $f(x) = \lim_{s \rightarrow \infty} f(x, s) =$ the least y such that $(\forall z) \theta(x, y, z)$.

Define a coloring $C: \mathbb{N} \rightarrow 2^a$ as follows. Let $D: \mathbb{N} \times \mathbb{N} \rightarrow 2$ be the function of Lemma 5.6 and set $C(s) = \langle D(f(x, s), s) : x < a \rangle$. By (3) there exists d such that $gs(H) \leq d$ for arbitrarily large C -homogeneous sets H . We claim that d is as desired. Let $x < a$ and suppose that $f(x)$ exists. Fix s such that $f(x, t) = f(x)$ for all $t > s$. We aim to prove that $D(f(x), \cdot)$ has arbitrarily large homogeneous sets H with $gs(H) \leq d$ and therefore it must be $f(x) < d$. Let k be given. By the assumption, there exists a C -homogeneous set G of size $s + k + 1$ and $gs(G) \leq d$. Let H consist of the last k elements of G . Then $|H| = k$, $gs(H) \leq d$ and $s < t$ for all $t \in H$. Suppose G is C -homogeneous for color i . Then for all $t \in H$ we have that $i(x) = C(t)(x) = D(f(x, t), t) = D(f(x), t)$ and hence H is homogeneous for $D(f(x), \cdot)$.

It remains to show that (3) implies IS_1^0 over RCA_0^* . We can apply the argument above to prove SII_1^0 , that is:

$$(\forall a)(\exists d)(\forall x < a)(\exists y \theta(x, y) \rightarrow (\exists y < d) \theta(x, y)),$$

where θ is Σ_0^0 . Define $f(x, s)$ to be the least $y < s$ such that $\theta(x, y)$ if y exists and s otherwise. Now Σ_0^0 -induction is sufficient to show that if $\exists y \theta(x, y)$ then $f(x) = \lim_{s \rightarrow \infty} f(x, s) =$ the least y such that $\theta(x, y)$. Define C as before by using D and show that d is as desired. \square

6. FINITE BROWN'S LEMMA

Theorem 6.1 (Brown's Lemma, finite). *Let $f: \mathbb{N} \rightarrow \mathbb{N}$. Then for all $r > 0$ there exists n such that every coloring $C: n \rightarrow r$ has a C -homogeneous set H with $|H| > f(gs(H))$.*

Proof. From Brown's Lemma by using König's Lemma. \square

Theorem 6.2. *The finite version of Brown's Lemma is provable in RCA_0 .*

Proof. The proof of [LR04, Theorem 10.33] can be formalized in RCA_0 . Let $f: \mathbb{N} \rightarrow \mathbb{N}$ be given. Without loss of generality we may assume that f is nondecreasing, for otherwise we can define $g(n) = \sum_{i \leq n} f(i)$. The proof is by Σ_1^0 -induction on r .

For $r = 1$, let $n_1 = f(1) + 2$. Suppose n_r works for r . We claim that $n_{r+1} = (r+1)f(n_r) + 1$ works for $r + 1$. Let $C: n_{r+1} \rightarrow r + 1$ be any coloring. Set $H_i = \{x < n_{r+1} : C(x) = i\}$ for $i \leq r$. We may assume that $|H_i| \leq f(gs(H_i))$ for every $i \leq r$, for otherwise we are done. We may also assume that $gs(H_i) \leq n_r$ for every $i \leq r$, for otherwise there exists an interval of size n_r which avoids some color i , and the conclusion follows by induction. As f is nondecreasing, we have that $f(gs(H_i)) \leq f(n_r)$ for all $i \leq r$. Therefore, $|H_i| \leq f(n_r)$ for every $i \leq r$, and

$$n_{r+1} = \sum_{i \leq r} |H_i| \leq (r+1)f(n_r),$$

a contradiction. \square

We define Brown's numbers as follows.

Definition 6.3. For $f: \mathbb{N} \rightarrow \mathbb{N}$ and $r > 0$ let $B_f(r)$ be the least natural number n such that every r -coloring of n has a homogeneous set H with $|H| > f(gs(H))$.

Remember that the superexponential function $2_k(n)$ is defined by $2_0(n) = n$ and $2_{k+1}(n) = 2^{2_k(n)}$. Let $2_r = 2_r(1)$. The proof of Theorem 6.2 gives superexponential upper bounds n_r for $B_f(r)$. For instance, if $f(d) = 2^d$, then $n_r \geq 2_r$.

Theorem 6.4 (Ardal [Ard10]). *Let $f(d) = md$ with $m > 0$. Then for all $r > 0$, $B_f(r) \leq r(2^{mr} - mr) + 1$.*

Recall that, for $f: \mathbb{N} \rightarrow \mathbb{N}$, BL_f is the statement “For all $r > 0$ there exists n such that every $C: n \rightarrow r$ has a C -homogeneous set H such that $|H| > f(gs(H))$.”

Theorem 6.5. $\text{EFA} \vdash (\forall m > 0) \text{BL}_{d \rightarrow md}$.

Proof. The reader can check that the proofs of Lemma 10, 11, 12, 13, 14, 22, 23 and Theorem 15, 24 of [Ard10] can be formalized in EFA . \square

We aim to show that RCA_0^* does not prove the finite version of Brown's lemma by proving that $B_f(r)$ is not elementary recursive for $f(d) = 2^d$.

Proposition 6.6. *The function $g(n) = 2_{\log_2 n}$ is not elementary recursive.*

Proof. Suppose g is elementary recursive. Then there exists a number k such that $2_{\log_2 n} < 2_k(n)$ for all n . Let $n = 2_{k+2}$. Then $2_{\log_2 n} = 2_{2_{k+1}}$ and $2_k(n) = 2_{2_{k+2}}$. On the other hand $2_{k+1} \geq 2k + 2$ for all k , and hence $2_{2_{k+1}} \geq 2_{2k+2}$, a contradiction. \square

Theorem 6.7. *Let $f(d) = 2^d$. Then for all $r > 0$, $B_f(r) > 2_{\log_2(r)}$.*

Proof. For all $s \geq 0$ we define a number $n_s \geq 2_s$ and a coloring $C_s: n_s \rightarrow 2^s$ witnessing $B(2^s) > n_s$.

In general, for $f: \mathbb{N} \rightarrow \mathbb{N}$ nondecreasing, $B_f(r) > n$ iff there exists a coloring $C: n \rightarrow r$ such that for all $i < r$ the finite set $H = \{x < n: C(x) = i\}$ satisfies

$$(*) \quad |I| \leq f(gs(I)) \text{ for every interval } I \text{ of } H.$$

Notice that $(*)$ is shift invariant. For finite sets $A < B$ let $d(A, B) = \min B - \max A$. Note that if A satisfies $(*)$ and $m \cdot |A| \leq f(d)$, then the set $H = \bigcup_{l < m} A_l$ satisfies $(*)$, where $A_0 < A_1 < \dots < A_{m-1}$, each A_l is a shift of A and $d(A_l, A_{l+1}) = d$.

We ensure that for all s and for all $i < 2^s$ the set $\{x < n_s: C_s(x) = i\}$ satisfies $(*)$ with $f(d) = 2^d$. For $s = 0$, let $n_0 = 2$ and $C_0 = 00$. For $s = 1$, let $n_1 = 2^4 = 16$ and

$$C_1 = 0011001100110011.$$

Suppose we have defined n_s and $C_s: n_s \rightarrow 2^s$. Let $n_{s+1} = 2^{s+1} \cdot 2^{n_0+n_1+\dots+n_s+1}$. Define $C_{s+1}: n_{s+1} \rightarrow 2^{s+1}$ by

$$C_{s+1} = C_s D_s C_s D_s \dots C_s D_s,$$

where $C_s D_s$ is repeated 2^{n_s} times and D_s is a copy of C_s with the color $i < 2^s$ replaced by $i + 2^s$. By induction one can show that $n_{s+1} = 2n_s 2^{n_s}$.

Claim. For all s and for all $i < 2^s$ the set $H_{i,s} = \{x < n_s: C(x) = i\}$ satisfies $(*)$.

By induction on s we prove that for all $i < 2^s$:

- (1) $|H_{i,s}| = n_s/2^s$;
- (2) $H_{i,s}$ satisfies $(*)$;
- (3) $n_s = \max H_{i,s} - \min H_{i,s} + n_0 + \dots + n_{s-1} + 1$.

For $s = 0$ this is true. Suppose it is true for s and let $m = 2^{n_s}$. Fix $i < 2^{s+1}$. By construction either $i < 2^s$ and $H_{i,s+1} = \bigcup_{l < m} A_l$, where $A_0 < A_1 < \dots < A_{m-1}$ and $A_l = H_{i,s} + 2ln_s$, or $i = j + 2^s$ with $j < 2^s$ and $H_{i,s+1} = \bigcup_{l < m} A_l$, where $A_0 < A_1 < \dots < A_{m-1}$ and $A_l = H_{j,s} + (2l+1)n_s$. Suppose $i < 2^s$. The argument for $i = j + 2^s$ is similar. Then

$$|H_{i,s+1}| = m \cdot |H_{i,s}| \stackrel{(1)}{=} m \cdot \frac{n_s}{2^s} = 2^{n_s} \cdot \frac{n_s}{2^s} = \frac{n_{s+1}}{2^{s+1}}.$$

We aim to show that $H_{i,s+1}$ satisfies $(*)$. By induction $H_{i,s}$ satisfies $(*)$ and so every A_l satisfies $(*)$ because $(*)$ is invariant under shift. Since $m \cdot |A_0| = |H_{i,s+1}| = 2^{n_s} \cdot \frac{n_s}{2^s} = 2^{n_0+\dots+n_s+1}$, it is sufficient to show that $d(A_l, A_{l+1}) = d(A_0, A_1) = n_0 + n_1 + \dots + n_s + 1$. Now

$$\begin{aligned} d(A_0, A_1) &= n_s + \min H_{i,s} + (n_s - \max H_{i,s}) \stackrel{(3)}{=} \\ &= n_s + (n_0 + \dots + n_{s-1} + 1) = n_0 + \dots + n_s + 1. \end{aligned}$$

Let $a = \min H_{i,s+1}$ and $b = \max H_{i,s+1}$. It remains to show that $n_{s+1} = b - a + n_0 + \dots + n_s + 1$. Write n_{s+1} as $a + (b - a) + (n_{s+1} - b)$. Now

$$a = \min H_{i,s+1} = \min H_{i,s}$$

and

$$n_{s+1} - b = n_{s+1} - \max H_{i,s+1} = (n_s - \max H_{i,s}) + n_s.$$

It follows that

$$\begin{aligned} n_{s+1} &= \min H_{i,s} + (b - a) + n_s - \max H_{i,s} + n_s = \\ &= b - a + n_s + (n_s + \min H_{i,s} - \max H_{i,s}) \stackrel{(3)}{=} b - a + n_s + (n_0 + \dots + n_{s-1} + 1) = \\ &= b - a + n_0 + \dots + n_s + 1. \end{aligned}$$

This completes the proof of the claim. By induction it is easy to prove that $n_s \geq 2_s$. It follows that

$$B(r) \geq B(2^{\log_2(r)}) > n_{\log_2(r)} \geq 2_{\log_2(r)},$$

as desired. \square

Notice that Theorem 6.7 is provable in $\text{I}\Sigma_1$.

Corollary 6.8. $\text{EFA} \not\vdash \text{BL}_{d \rightarrow 2^d}$. In particular, RCA_0^* does not prove the finite version of Brown's lemma.

Proof. Let $f(d) = 2^d$. The function $B_f(r)$ is Σ_0 -definable and so $\text{EFA} \vdash \text{BL}_f$ iff $B_f(r)$ is provably recursive in EFA iff $B_f(r)$ is elementary recursive. By Theorem 6.7 the function $B_f(r)$ is not elementary recursive. Therefore $\text{EFA} \not\vdash \text{BL}_f$. The second part follows from the fact that the statement BL_f is Π_2 and RCA_0^* is conservative over EFA for Π_2 -sentences. \square

As the finite van der Waerden's theorem is already provable in RCA_0 and presumably in RCA_0^* , the question whether Brown's lemma implies van der Waerden's theorem can be settled only over a very weak system of arithmetic.

Question. What is the relationship between (the finite versions of) Brown's lemma and van der Waerden's theorem over a suitable bounded fragment of second-order arithmetic?

APPENDIX A.

The classical proof of Brown's lemma (see for instance [Bro71, Lemma 1] or [LR04, Theorem 10.32]) is based on the following fact.

Fact A.1. Let $r \in \mathbb{N}$. If $S \subseteq r^{<\mathbb{N}}$ is infinite, then there exists $g: \mathbb{N} \rightarrow r$ such that for all k there is $\sigma \in S$ with $g \upharpoonright k \subseteq \sigma$.

Classical proof of Brown's lemma. By induction on r . If $r = 1$, there is nothing to prove. Let $C: \mathbb{N} \rightarrow r + 1$. We say that $\sigma \in r^{<\mathbb{N}}$ is a factor of C , and write $\sigma \in S(C)$, if $\sigma \subseteq \langle C(y), C(y+1), C(y+2), \dots \rangle$ for some y . Let S be the set of factors which avoid the color r , that is $S = S(C) \cap r^{<\mathbb{N}}$.

We may assume that $\{x \in \mathbb{N}: C(x) = r\}$ is not piecewise syndetic, otherwise we are done. We claim that for every d there is $\sigma \in S$ of length d . Suppose not. Let d be such that every $\sigma \in S(f)$ of length d does not avoid r . Then it is easy to see that $\{x \in \mathbb{N}: C(x) = r\}$ is piecewise d -syndetic, contrary to our assumption. Therefore S is infinite.

By Fact A.1, there exists $g: \mathbb{N} \rightarrow r$ such that for all n there exists $\sigma \in S$ with $g \upharpoonright n \subseteq \sigma$. By induction, let $i < r$ such that $\{x \in \mathbb{N}: g(x) = i\}$ is piecewise syndetic and let d be a witness. We claim that $\{x \in \mathbb{N}: C(x) = i\}$ is piecewise d -syndetic. Fix n . There exists a set F of size n and gaps bounded by d such that $g(x) = i$ for all $x \in F$. Let $k > F$. By the assumption on g , let $\sigma \in S$ such that $g \upharpoonright k \subseteq \sigma$. Now, σ is a factor of C , say $\sigma \subseteq \langle C(y), C(y+1), \dots \rangle$. Hence, for all $x < k$ we have that $g(x) = C(y+x)$. It easily follows that $y + F$ is as desired (size n , gaps bounded by d , C -homogeneous for color i). \square

A loose formalization of the above proof goes through ACA_0 plus Π_1^1 -induction. In particular, Fact A.1 is provable in ACA_0 and actually equivalent to ACA_0 .

Proposition A.2. *Over RCA_0 , the following are equivalent:*

- (1) ACA_0
- (2) *Let $r \in \mathbb{N}$. If $S \subseteq r^{<\mathbb{N}}$ is infinite, then there exists $g: \mathbb{N} \rightarrow r$ such that for all n there is $\sigma \in S$ with $g \upharpoonright n \subseteq \sigma$.*
- (3) *If $S \subseteq 2^{<\mathbb{N}}$ is infinite, then there exists $g: \mathbb{N} \rightarrow 2$ such that for all n there is $\sigma \in S$ with $g \upharpoonright n \subseteq \sigma$.*

Proof. (1) implies (2). Define a tree $T \subseteq r^{<\mathbb{N}}$ by letting $\tau \in T$ iff $(\exists \sigma \in S)\tau \subseteq \sigma$. T is an infinite finitely branching tree. By König's Lemma, there exists an infinite path g . Then g is as desired. (2) implies (3) is trivial.

For (3) implies (2), let $f: \mathbb{N} \rightarrow \mathbb{N}$ be a one-to-one function. We aim to show that the range of f exists. Define $S \subseteq 2^{<\mathbb{N}}$ by letting $\sigma \in S$ if and only if $(\forall y < |\sigma|)(\sigma(y) = 1 \leftrightarrow (\exists x < |\sigma|)f(x) = y)$. Then S is infinite. By (2), let g be such that every initial segment of g is an initial segment of some $\sigma \in S$. We claim that $y \in \text{ran } f$ iff $g(y) = 1$. Suppose $g(y) = 1$ and let $g \upharpoonright y + 1 \subseteq \sigma$ with $\sigma \in S$. Then $\sigma(y) = 1$ and so $y \in \text{ran } f$. Suppose that $y \in \text{ran } f$ and let $f(x) = y$. Let $n > x, y$ and $\sigma \in S$ such that $g \upharpoonright n \subseteq \sigma$. Then $\sigma(y) = 1$ and so $g(y) = 1$. \square

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